Root finding over finite fields using Graeffe transforms

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Root finding over finite fields

Given $f \in \mathbb{F}_q[X]$, compute its roots, that is $\{\alpha \in \mathbb{F}_q : f(\alpha) = 0\}$.

Input size: $(1 + d) \log q$ where $d = \deg(f)$

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$$f(X) = \prod_{i=1}^{d}(X - \alpha_i), \quad \alpha_i \in \mathbb{F}_q^*, \quad \alpha_i \neq \alpha_j$$

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- Motivated by sparse interpolation

[van der Hoeven & Lecerf, 2014]

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- Better complexity bounds when $q-1$ is sufficiently *smooth* [Moenck (1977), von zur Gathen (1987), Mignotte-Schnorr (1988), Rónyai (1989), Shoup (1991, 1992), Źrałek (2010)]

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- FFT finite field: $p = M \cdot 2^m + 1$ with $M = O(\log p)$
  - Adapt old algorithms
  - New technique based on Graeffe transforms
  - Fast implementations

Adapt old algorithms

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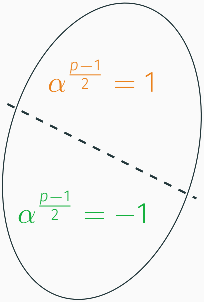

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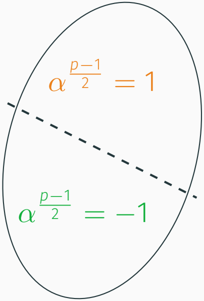

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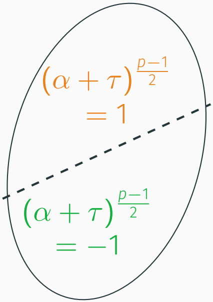

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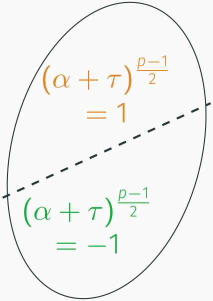

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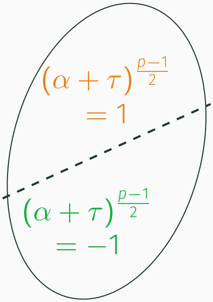

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Randomized algorithm

The roots of $f \in \mathbb{F}_p[X]$ can be computed in expected time $\tilde{O}(d \log^2 p)$.

Modified Rabin's algorithm (for FFT finite fields)

$$X^{p-1} - 1 = \prod_{i=0}^{2^\ell - 1} (X^{M2^{m-\ell}} - \xi^i), \text{ where } \xi \text{ is primitive of order } 2^\ell.$$

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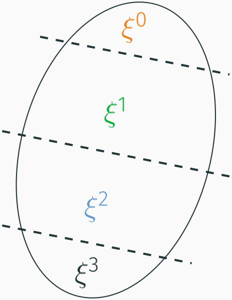

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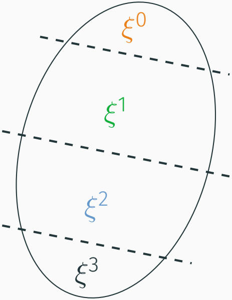

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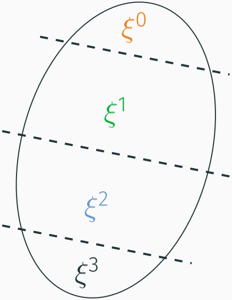

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Worthwhile in practice for **small** $\ell = 2, 3, \ldots$

New technique: Graeffe transform

Let $f(X) = \prod_i (X - \alpha_i) \in \mathbb{F}_p[X]$.

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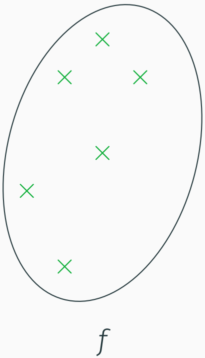
Remarks:

- $G_{\rho_1\rho_2} = G_{\rho_1} \circ G_{\rho_2}$, and in particular $G_{2^\ell} = G_2 \circ \cdots \circ G_2$
- $G_{p-1}(f)(X) = \prod_i (X - \alpha_i^{p-1}) = (X - 1)^d$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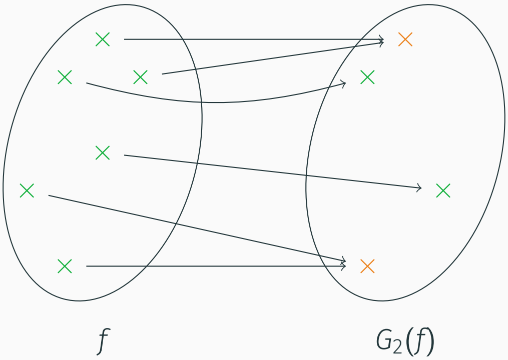

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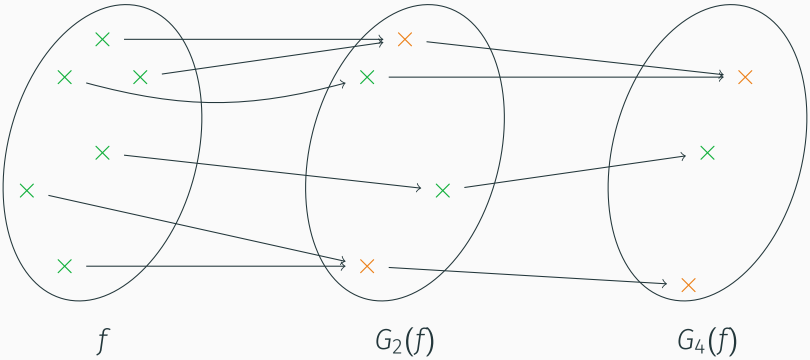

$$f \xrightarrow{G_2} g_1 \xrightarrow{G_2} g_2 \xrightarrow{G_2} \cdots \xrightarrow{G_2} g_m \xrightarrow{G_M} g_{m+1}$$

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  - Multiple roots: If $\beta = \zeta^e$,

$$\alpha_i, \alpha_j \in \{\zeta^{e/2}, \zeta^{(e+2^m M)/2}\}$$

Theorem

Given $f \in \mathbb{F}_q[X]$ satisfying (A), the irreducible factorization of $(q-1)$ and a primitive element of $\mathbb{F}_q^*$, the roots of f can be computed in time

$$\tilde{O}(\sqrt{S_1(q-1)}d \log^2 q) + (d \log^2 q)^{1+o(1)}$$

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- Refines Shoup's complexity bounds

Randomization

Definition

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Lemma

Let $g + \varepsilon \overline{g} = G_{2^\ell}(f + \varepsilon f')$. **A nonzero root β of g is simple iff $\overline{g}(\beta) \neq 0$.** The corresponding root of f is $\alpha = 2^\ell \beta g'(\beta)/\overline{g}(\beta)$.

Goal: Ensure many simple roots

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$$Z_0 \subset Z(f) \qquad Z_m \subset \{\xi^e : 0 \leq e < M\}$$

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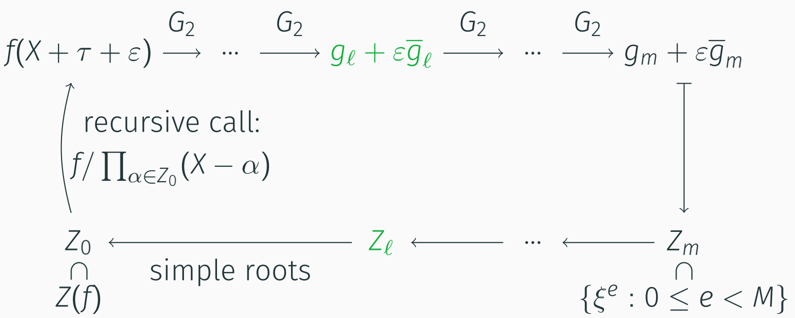

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- Same asymptotic as Rabin's algorithm
- Better efficiency in practice
- Primitive elements easy to compute in practice

Heuristic

If $2^\ell \simeq p/d$, $G_{2^\ell}(f(X+\tau))$ has **$\Omega(d)$ simple roots** with probability $\geq 1/2$, for a random $\tau \in \mathbb{F}_p$.

Justification: holds for a random f rather than $f(X+\tau)$.

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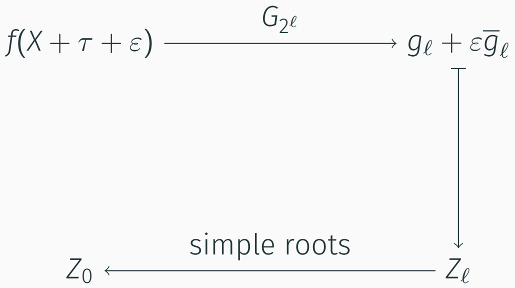
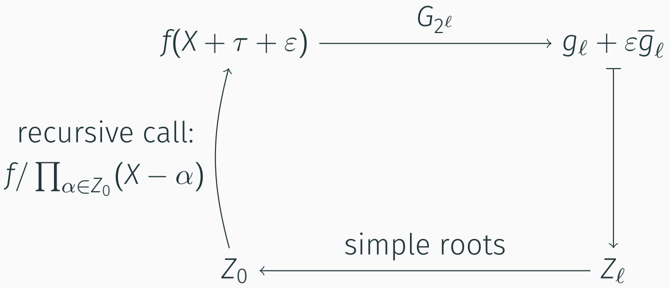


Heuristic algorithm

Heuristic

If $2^\ell \simeq p/d$, $G_{2^\ell}(f(X+\tau))$ has $\Omega(d)$ **simple roots** with probability $\geq 1/2$, for a random $\tau \in \mathbb{F}_p$.

Justification: holds for a random f rather than $f(X+\tau)$.

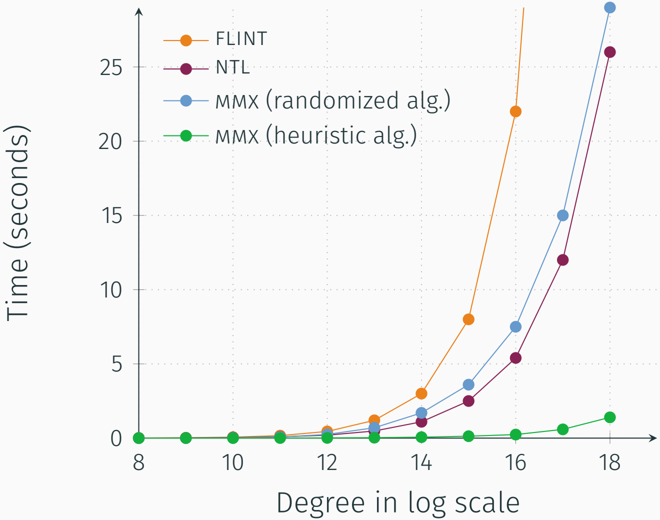


Theorem

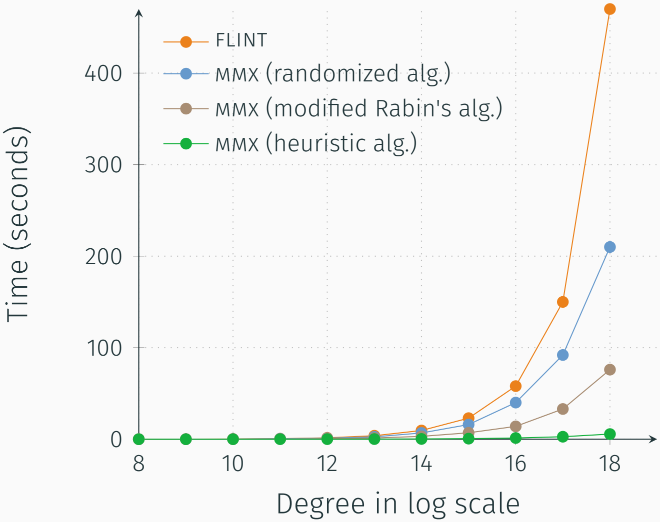
Suppose that f is chosen at random in $\mathbb{F}_p[X]$ or that the heuristic holds. Given a primitive element of $\mathbb{F}_p^*$, the heuristic algorithm runs in **expected time $\tilde{O}(d \log^2 p)$**, for $p = M \cdot 2^m + 1$ with $M = O(\log p)$.

Running times

$p = 7 \cdot 2^{26} + 1$


FLINT
NTL
MMX (randomized alg.)
MMX (heuristic alg.)
Time (seconds)
0
5
10
15
20
25
8
10
12
14
16
18
Degree in log scale

$$p = 5 \cdot 2^{55} + 1$$



FLINT
MMX (randomized alg.)
MMX (modified Rabin's alg.)
MMX (heuristic alg.)
Time (seconds)
400
300
200
100
0
8
10
12
14
16
18
Degree in log scale

- Revisit classical algorithms for FFT finite fields

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- New approach using Graeffe transforms
    - ✓ deterministic complexity bounds
    - ✓ probabilistic complexity bounds
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Merci de votre attention !